TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 183, September 1973

IRREDUCIBLE REPRESENTATIONS OF THE C^*-ALGEBRA GENERATED BY A QUASINORMAL OPERATOR

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ABSTRACT. For A a quasinormal operator on Hilbert space, we determine the irreducible representations of $C^*(A)$, the C^*-algebra generated by A and the identity. We also explicitly describe the topology on the space of unitary equivalence classes of irreducible representations of $C^*(A)$ and calculate the regularized transform of $C^*(A)$, thus exhibiting an isomorphic copy of $C^*(A)$.

1. **Introduction and preliminaries.** For A a bounded linear operator on a Hilbert space, let $C^*(A)$ denote the C^*-algebra generated by A and the identity I. By the spectrum of $C^*(A)$ we mean the set of unitary equivalence classes of irreducible representations of $C^*(A)$ equipped with the hull-kernel topology [8, § 3]. (By "representation" we mean an identity preserving *-representation.) A character of a C^*-algebra is a multiplicative linear functional of the algebra onto the complex numbers. In [6] the spectrum of $C^*(A)$ was calculated for the class of binormal operators introduced by A. Brown [3]. In this paper we calculate the spectrum of $C^*(A)$ for A a quasinormal operator. Quasinormal operators were introduced (under another name) by Brown in [2] and are operators A such that A commutes with A^*A.

For K a Hilbert space let $\tilde{K}$ be the set of sequences $(x_0, x_1, x_2, \cdots)$ with $x_i \in K$ for all i and the sequence $(\|x_i\|)$ square-summable. If P is a positive operator on K, define a bounded operator on $\tilde{K}$ by $\hat{P}(x_0, x_1, x_2, \cdots) = (0, Px_0, Px_1, \cdots)$. Then $\hat{P}$ is quasinormal. In [2] Brown proved that the most general quasinormal operator is unitarily equivalent to one of the form $\hat{P} \oplus N$, where P is positive and one-to-one and N is normal. It is then immediate that an irreducible quasinormal operator is either an operator on a one-dimensional space or is unitarily equivalent to a nonzero positive scalar multiple of U_+, where U_+ is the unilateral forward shift of multiplicity one. Thus if A is quasinormal and π is an irreducible representation of $C^*(A)$, then since $\pi(A)$ is irreducible quasinormal either π is a character of $C^*(A)$ or $\pi(A)$ is unitarily equivalent to

Received by the editors September 28, 1972.

AMS (MOS) subject classifications (1970). Primary 46L05, 46L25, 47C10; Secondary 47B20.

Key words and phrases. Irreducible representation, spectrum, quasinormal operator, Hausdorff compactification, regularized transform.

[1] The author was partially supported by an NSF grant.

a positive scalar multiple of U_+. In either case $C^*(\pi(A))$ contains the compact operators on H_π, the Hilbert space associated with π, hence $C^*(A)$ is a GCR or postliminaire C^*-algebra when A is quasinormal [8, §9]. In §2 of this paper we parametrize the spectrum of $C^*(A)$ and exhibit the topology on the spectrum. In §3 we calculate what J. M. G. Fell [9] calls the regularized transform algebra of $C^*(A)$, thus exhibiting an isomorphic copy of $C^*(A)$.

We note that L. A. Coburn determined the spectrum of $C^*(U_+)$ in [7]. The identity representation is the unique (up to unitary equivalence) infinite-dimensional irreducible representation of $C^*(U_+)$ and for each complex z, $|z| = 1$, there is a character $\hat{z}$ of $C^*(U_+)$ with $\hat{z}(U_+) = z$. These are all the equivalence classes of irreducible representations of $C^*(U_+)$. H. Behncke noted that $C^*(A)$ is GCR for A quasinormal in [1]. H. Gonshor essentially found the regularized transform algebra of $C^*(A)$ for A binormal in [11].

2. **The spectrum of $C^*(A)$.** Throughout this section A will denote a quasinormal operator. As shown in the introduction if π is an irreducible representation of $C^*(A)$, then either π is a character or $\pi(A)$ acts on separable infinite-dimensional Hilbert space. We note that a representation π of $C^*(A)$ is completely determined by $\pi(A)$. We first describe some irreducible representations of $C^*(A)$ and then show that every irreducible representation is unitarily equivalent to one of these.

As shown in the introduction, we can write $A = \hat{P} \oplus N$, where P is positive and N is normal (we do not assume that P is one-to-one). We denote the spectrum of an operator T by $\mathrm{sp}(T)$ and the approximate point spectrum of T by $a(T)$. By [5, Proposition 8] $a(\hat{P}) = \{z\alpha : |z| = 1, \alpha \in \mathrm{sp}(P)\}$. Now $\hat{P}$ is hyponormal (i.e., $\hat{P}(\hat{P})^* \leq (\hat{P})^*\hat{P}$) so, by [4, Corollary 10], for each $\lambda \in a(\hat{P})$ there is a character f of $C^*(\hat{P})$ such that $f(\hat{P}) = \lambda$. But $C^*(A)$ is contained in $C^*(\hat{P}) \oplus C^*(N)$ and π defined by $\pi(B \oplus D) = f(B)$ for $B \in C^*(\hat{P})$ and $D \in C^*(N)$ is clearly a character of $C^*(\hat{P}) \oplus C^*(N)$, so for each $\lambda \in a(\hat{P})$ there is a character π of $C^*(A)$ with $\pi(A) = \lambda$. Similarly, if $\lambda \in \mathrm{sp}(N)$ then by the Gel'fand theory there is a character f of $C^*(N)$ with $f(N) = \lambda$, and there is a character π of $C^*(A)$ with $\pi(A) = \lambda$.

Now let λ be a nonzero number in $\mathrm{sp}(P)$. We show that there is an irreducible representation π of $C^*(A)$ with $\pi(A) = \lambda U_+$. If P acts on a Hilbert space K and $\hat{P}$ acts on $\hat{K}$, then it is easily seen that $\hat{P}$ is unitarily equivalent to the operator $P \otimes U_+$ acting on $K \otimes l^2$. Then $C^*(\hat{P})$ is contained in the tensor product C^*-algebra $C^*(P) \overset{*}{\otimes} C^*(U_+)$ where the norm in $C^*(P) \overset{*}{\otimes} C^*(U_+)$ is the operator norm as operators on $K \otimes l^2$. Then if $0 \neq \lambda \in \mathrm{sp}(P)$ there is a character ρ of $C^*(P)$ with $\rho(P) = \lambda$. Let θ be the identity representation of $C^*(U_+)$. Then $\rho \otimes \theta$, defined on the algebraic tensor product $C^*(P) \otimes C^*(U_+)$ by $(\rho \otimes \theta)(B \otimes D) = \rho(B)\,\theta(D)$ for all $B \in C^*(P)$, $D \in C^*(U_+)$, extends to an irreducible representation of $C^*(P) \overset{*}{\otimes} C^*(I_+)$ [12]. Thus there is a representation π of $C^*(\hat{P})$ with $\pi(\hat{P}) = \lambda U^+$. Since $C^*(A)$ is contained in $C^*(\hat{P}) \oplus C^*(N)$, it is clear that there is an irreducible representation π of $C^*(A)$ with $\pi(A) = \lambda U_+$.

We now show that the rerpesentations of $C^*(A)$ described above are all the irreducible representations of $C^*(A)$.

Theorem 1. *Let A be a quasinormal operator with decomposiion $A = \hat{P} \oplus N$, where P is positive and N is normal. Then for each $\lambda \in a(\hat{P}) \cup \mathrm{sp}(N)$ there is a character π of $C^*(A)$ with $\pi(A) = \lambda$. If $0 \neq \lambda \in \mathrm{sp}(P)$ then there is a representation π of $C^*(A)$ with $\pi(A) = \lambda U_+$. Conversely, every irreducible representation of $C^*(A)$ is unitarily equivalent to one of these.*

Proof. Let π be an irreducible representation of $C^*(A)$ on a Hilbert space H. Then since $C^*(A) \subseteq C^*(\hat{P}) \oplus C^*(N)$, there is a Hilbert space H_1 containing H as a subspace and an irreducible representation π_1 of $C^*(\hat{P}) \oplus C^*(N)$ on H_1 such that $\pi_1(D)|H = \pi(D)$ for each $D \in C^*(A)$ [8, 2.10.2]. Now $I \oplus 0$ and $0 \oplus I$ are in the center of $C^*(\hat{P}) \oplus C^*(N)$, so since π_1 is irreducible either $\pi_1(I \oplus 0) = 0$ or $\pi_1(0 \oplus I) = 0$.

First, assume that $\pi_1(I \oplus 0) = 0$. Then $\pi_1(D \oplus 0) = 0$ for all $D \in C^*(\hat{P})$, so $\pi_1(C^*(\hat{P}) \oplus C^*(N))$ is irreducible and abelian, hence H_1 is one-dimensional and $H = H_1$. Then the map $B \to \pi_1(0 \oplus B)$ is a character of $C^*(N)$, so $\pi(A) = \pi_1(\hat{P} \oplus N) = \pi_1(0 \oplus N)$ is in $\mathrm{sp}(N)$. Thus π is a representation of one of the forms mentioned in the theorem.

Now assume that $\pi_1(0 \oplus I) = 0$. Then π_2 defined on $C^*(\hat{P})$ by $\pi_2(B) = \pi_1(B \oplus 0)$ is an irreducible representation of $C^*(\hat{P})$. Thus $\pi_2(\hat{P})$ is an irreducible quasinormal so $\pi_2(\hat{P})$ is either a positive scalar multiple of U_+ or H_1 is a one-dimensional space. If $H = H_1$ is one-dimensional and $\pi_2(\hat{P}) = \lambda$, then π_2 is a character on $C^*(\hat{P})$, so by [4, Proposition 8], $\lambda = \pi_2(\hat{P}) = \pi(A)$ is in $a(\hat{P})$ and π is a character of $C^*(A)$, and π is a representation of one of the forms mentioned in the theorem. If $\pi_2(\hat{P}) = \lambda U_+$ for $\lambda > 0$, then since $a(\pi_2(\hat{P})) = \{\lambda z : |z| = 1\}$, $a(\hat{P}) = \{\alpha z : \alpha \in \mathrm{sp}(P),\ |z| = 1\}$, and $a(\pi_2(\hat{P})) \subseteq a(\hat{P})$ by [4, Corollary 5], we have that $\lambda \in \mathrm{sp}(P)$. Finally, $\pi_2(\hat{P}) = \pi_1(\hat{P} \oplus N) = \pi_1(A)$ and $\pi_1(A)$ acts irreducibly on H_1, but H reduces $\pi_1(A)$, hence $H = H_1$ and $\pi(A) = \lambda U_+$. Hence, in any case, π must be of one of the forms mentioned in the theorem.

Now let $T_1 = \{(z, 0) : z \in a(\hat{P}) \cup \mathrm{sp}(N)\}$ and $T_2 = \{(0, x) : 0 \neq x \in \mathrm{sp}(P)\}$. Then let $T = T_1 \cup T_2$ and give T the following topology: call a subset F of T closed if and only if $F \cap T_1$ and $F \cap T_2$ are closed in their natural topologies and if $x \in F \cap T_2$, then $zx \in F \cap T_1$ for all complex z of modulus one. This is easily seen to be a compact T_0 topology on T. Let X denote the spectrum of $C^*(A)$ and using Theorem 1 define a bijection θ from T onto X by $\theta(z, 0)(A) = z$ if

$(z, 0) \in T_1$ and $\theta(0, x)(A) = xU_+$ if $(0, x) \in T_2$. Here we mean that $\theta(z, 0)$ is a character on $C^*(A)$ which takes A to z, and $\theta(0, x)$ is a representation of $C^*(A)$ on l^2 which takes A to xU_+.

Theorem 2. *Let A be quasinormal. Then the map $\theta: T \to X$ described above is a homeomorphism of the set T and the spectrum of $C^*(A)$.*

Proof. Recall [8, 3.3.3] that if $\{D_i\}$ is a dense subset of $C^*(A)$, then a base for the hull-kernel topology on X is given by the sets $U_i = \{\pi \in X: \|\pi(D_i)\| > 1\}$. We will use the dense subset of $C^*(A)$ consisting of operators of the form $D = p(A, A^*)$ where p is a polynomial in two noncommuting variables. Since T and X both satisfy the second axiom of countability, sequential convergence determines the topology, so we do not need to consider nets. We show that θ is continuous. Let $t \in T$ and let a sequence (t_n) converge to t. Now $\{(z, 0): z \in \mathrm{sp}(N), z \notin a(\hat{P})\}$ and T_2 are both open in T. If $t \in T_2$ we may assume that all $t_n \in T_2$. Then if $t = (0, x)$, $t_n = (0, x_n)$, we have that x_n converges to x as a sequence of real numbers. Hence $\theta(t_n)(p(A, A^*)) = p(x_n U_+, x_n U_+^*)$ which converges to $p(xU_+, xU_+^*) = \theta(t)(p(A, A^*))$ in the norm as bounded operators on l^2. Hence $\theta(t_n)$ converges to $\theta(t)$ in X. Likewise, if $t \in \{(z, 0): z \in \mathrm{sp}(N), z \notin a(\hat{P})\}$, then we may assume that $t_n \in \{(z, 0): z \in \mathrm{sp}(N), z \notin a(\hat{P})\}$ and the sequence of complex numbers $\theta(t_n)(p(A, A^*))$ will converge to $\theta(t)(p(A, A^*))$, so that $\theta(t_n)$ converges to $\theta(T)$ in X. Suppose now that $t = (z, 0)$ with $z \in a(\hat{P})$. If $z \neq 0$, let $(z/|z|)^\wedge$ be the character on $C^*(U_+)$ which takes U_+ to $(z/|z|)$. Then

$$|(z/|z|)^\wedge(p(|z|U_+, |z|U_+^*))| = |p(z, \bar{z})| \le \|p(|z|U_+, |z|U_+^*)\|.$$

If $z = 0$, then we clearly have $|p(z, \bar{z})| \le \|p(|z|U_+, |z|U_+^*)\|$. Now let (t_n) be a sequence in T which converges to $t = (z, 0)$, $z \in a(\hat{P})$, and suppose that the $t_n = (0, x_n)$ are in T_2. Then clearly x_n converges to $|z|$ in the natural topology on $\mathrm{sp}(P)$, and we have $\|\theta(t_n)(p(A, A^*))\| = \|p(x_n U_+, x_n U_+^*)\|$ which converges to $\|p(|z|U_+, |z|U_+^*)\|$. So if $\|\theta(t)(p(A, A^*))\| = |p(z, \bar{z})| > 1$, then $\|\theta(t_n)(p(A, A^*))\|$ is greater than one for large n. If the sequence (t_n) is in T_1, then clearly $\theta(t_n)(p(A, A^*))$ converges to $\theta(t)(p(A, A^*))$. We have thus shown that if (t_n) converges to t in T, then $\theta(t_n)$ converges to $\theta(t)$ in X.

To show that θ is a homeomorphism we take a closed set $F \subseteq T$ and show that $\theta(F)$ is closed in X. Now the set of characters of $C^*(A)$ is easily seen to be a compact Hausdorff subspace of X. We have $\theta(F) = \theta(F \cap T_1) \cup \theta(F \cap T_2)$, and $F \cap T_1$ is a closed, hence compact, subset of T_1; thus $\theta(F \cap T_1)$ is a compact subspace of the characters of $C^*(A)$ and $\theta(F \cap T_1)$ is closed in X. So let $\pi_n = \theta(0, x_n)$ be a sequence in $\theta(F \cap T_2)$ which converges in X to $\pi = \theta(t)$. We need only show that $t \in F$. There is a subsequence of x_n which converges to

$y \in \mathrm{sp}(P)$, so we may assume that x_n converges to y. First suppose that $t = (0, x) \in T_2$. Then since for each $B \in C^*(A)$ the function $\pi \longrightarrow \|\pi(B)\|$ is lower semicontinuous on X [8, 3.3.2], we have that $\|\pi(A)\| \leq \liminf \|\pi_n(A)\|$, or $\|xU_+\| \leq \liminf \|x_n U_+\|$, hence $x \leq y$. Also if $B = \|P\| - (A^*A)^{1/2}$, then $\|\pi(B)\| \leq \liminf \|\pi_n(B)\|$ or $\|(\|P\| - xI)\| \leq \liminf \| \|P\| - x_n I\|$ so $y \leq x$ and $y = x$. But F is closed and $(0, x_n)$ is in F, so $(0, x) \in F$. Now suppose that $t = (z, 0)$ is in T_1. Then almost exactly as in the case when $t \in T_2$ we see that $|z| \leq y$ and $\| \|P\| - |z|I\| \leq \| \|P\| - yI\|$ so that $y = |z|$. But $x_n \in F \cap T_2$ and x_n converges to y, so that $t = (z, 0) \in F$ by the definition of closed sets in T. So θ is a homeomorphism.

3. **The regularized transform of** $C^*(A)$. If A is a bounded operator on a Hilbert space, then A can be well understood without the structure of $C^*(A)$ being understood. It usually will not be clear which bounded operators are in $C^*(A)$. In this section we exhibit a C^*-algebra which is isomorphic to $C^*(A)$, for A a quasinormal operator. To do this we use a Stone-Weierstrass type theorem of Fell's [9, Theorem 1.4]. We thus need to sketch part of [9].

Let S be a compact Hausdorff space (Fell only assumes locally compact Hausdorff). For each $s \in S$ let $\mathfrak{A}(s)$ be a C^*-algebra. A full algebra of operator fields on S is a family $\mathfrak{A}$ of functions on S, $A(s) \in \mathfrak{A}(s)$ for each $s \in S$ and $A \in \mathfrak{A}$ satisfying:

(1) $\mathfrak{A}$ is a *-algebra under the pointwise algebraic operations;

(2) for each $A \in \mathfrak{A}$ the function $s \longrightarrow \|A(s)\|$ is continuous on S;

(3) for each s, $\{A(s);\ A \in \mathfrak{A}\} = \mathfrak{A}(s)$;

(4) $\mathfrak{A}$ is complete in the norm $\|A\| = \sup\{\|A(s)\| : s \in S\}$.

The algebra $\mathfrak{A}(s)$ is called the component of $\mathfrak{A}$ at s. A function f defined on S with $f(s) \in \mathfrak{A}(s)$ is continuous (with respect to $\mathfrak{A}$) at s_0, if for each $\epsilon > 0$, there is an element $A \in \mathfrak{A}$ and a neighborhood U of s_0 such that $\|f(s) - A(s)\| < \epsilon$ for all $s \in U$. We say that f is continuous on S if it is continuous at all points of S. The algebra $\mathfrak{A}$ is called a maximal full algebra of operator fields if any f which is continuous with respect to $\mathfrak{A}$ on S is actually in $\mathfrak{A}$.

Let $\mathfrak{B}$ and $\mathfrak{C}$ be C^*-algebras. A $(\mathfrak{B}, \mathfrak{C})$ correlation is a relation R contained in $\mathfrak{B} \times \mathfrak{C}$ such that, for some third C^*-algebra $\mathfrak{D}$ and some *-homomorphisms f and g of $\mathfrak{B}$ and $\mathfrak{C}$ respectively *onto* $\mathfrak{D}$, we have $b\ R\ c$ if and only if $f(b) = g(c)$. Let $\mathfrak{A}$ be a maximal full algebra of operator fields on S. If r and s are distinct points of S and R is an $(\mathfrak{A}(r), \mathfrak{A}(s))$ correlation, we define $\mathfrak{A}(r, s; R) = \{A \in \mathfrak{A}: A(r)\ R\ A(s)\}$. Clearly $\mathfrak{A}(r, s; R)$ is a C^*-algebra. Let $\mathfrak{A}_0$ be any full algebra of operator fields contained in $\mathfrak{A}$. Then Fell's Stone-Weierstrass theorem says that $\mathfrak{A}_0$ is the intersection of those $\mathfrak{A}(r, s; R)$ which contain $\mathfrak{A}_0$.

We now sketch the construction of Fell's regularized transform of a C^*-algebra

$\mathfrak{A}$ [9, § II]. Let X be any locally compact space with no separation axioms assumed. Let $\mathcal{C}(X)$ be the set of closed subsets of X. For each compact subset C of X and each finite family $\mathcal{F}$ of nonvoid open subsets of X, let $U(C, \mathcal{F})$ be the set of all Y in $\mathcal{C}(X)$ such that $Y \cap C = \emptyset$ and $Y \cap B \neq \emptyset$ for each $B \in \mathcal{F}$. The set of all such $U(C, \mathcal{F})$ forms a basis for the open sets of a compact Hausdorff topology on $\mathcal{C}(X)$ called the H-topology. Then $H(X)$ is defined as the closure in $\mathcal{C}(X)$ of the family of all closures of one-element subsets of X, and $H(X)$ is called the Hausdorff compactification of X. By the limit set of a net (x_α) in X we mean the set of those y in X such that (x_α) converges to y. The net (x_α) is primitive if the limit set of (x_α) is the same as the limit set of each subnet of (x_α). It is proved in [10] that the elements of $H(X)$ are exactly those closed subsets of X which are the limit set of some primitive net of elements of X. The details of the Hausdorff compactification of X are in [10].

If $\mathfrak{A}$ is a C^*-algebra then the spectrum $\hat{\mathfrak{A}}$ of $\mathfrak{A}$ is a locally compact space [8, Theorem 2.1]. Let $H(\hat{\mathfrak{A}})$ denote the Hausdorff compactification of $\hat{\mathfrak{A}}$. If $Y \in H(\hat{\mathfrak{A}})$ let $\mathfrak{A}(Y)$ equal $\mathfrak{A}/I(Y)$, where $I(Y) = \bigcap\{\pi^{-1}(0): \pi \in Y\}$. For $a \in \mathfrak{A}$ let $\tilde{a}$ be the function with domain $H(\hat{\mathfrak{A}})$ and range the union of the $\mathfrak{A}(Y)$, $Y \in H(\hat{\mathfrak{A}})$, defined by $\tilde{a}(Y) = a + I(Y)$ (the coset of a in $\mathfrak{A}/I(Y)$). The family of all $\tilde{a}$, $a \in \mathfrak{A}$ is called the regularized transform of $\mathfrak{A}$. The regularized transform of $\mathfrak{A}$ is a full algebra of operator fields on $H(\hat{\mathfrak{A}})$ and is a C^*-algebra which is isomorphic to $\mathfrak{A}$.

In this section we calculate the regularized transform of $C^*(A)$, for A a quasinormal operator. By Theorem 2, we can identify the topological space T with the spectrum of $C^*(A)$. If Y is a closed subset of T which is the limit set of some primitive net of elements of T, then some thought shows that in fact Y is the closure of some one-element subset of T. Hence elements of $H(T)$ are just the closures of one-element subsets of T. Now points in T_1 are closed in T and if $t = (0, x) \in T_2$, then the closure of t is $\{t\} \cup \{(zx, 0): |z| = 1\}$. We then identify the closure of t with the point $(0, x)$ in $R^2 \times R$. Hence, as a set, it is clear that $H(T)$ can be identified with the set $Q = Q_1 \cup Q_2$ in $R^2 \times R$, where $Q_1 = \{(z, 0): z \in a(\hat{P}) \cup \text{sp}(N)\}$ and $Q_2 = \{(0, x): 0 \neq x \in \text{sp}(P)\}$. It is routine (but tedious to write down) that a subset of Q which is open in the relative topology from $R^2 \times R$ is open when considered as a subset of $H(T)$ is the relative H-topology. Hence the natural bijection from $H(T)$ to Q is continuous and hence is a homeomorphism, since $H(T)$ and Q are both compact Hausdorff spaces. We thus identify the spaces $H(T)$ and Q.

The component algebras over each point in Q are easily described. If z is in $a(\hat{p}) \cup \text{sp}(N)$, then the component algebra over $(z, 0)$ is just the complex numbers. If $0 \neq x \in \text{sp}(P)$ then the component algebra over $(0, x)$ is $C^*(U_+)$. Thus all the component algebras are primitive. This is to be contrasted with the CCR case: If $\mathfrak{R}$ is a CCR C^*-algebra with every component algebra primitive, then the spectrum of $\mathfrak{R}$ must be Hausdorff [9, p. 243]. If $B \in C^*(A)$ then the regularized transform $\widetilde{B}$ of B is a complex-valued function on Q_1 and takes values in $C^*(U_+)$ on Q_2. If θ is the function defined just before Theorem 2, then clearly $\widetilde{B}(q) = \theta(q)(B)$ for each $q \in Q$. We now show exactly which functions occur in the regularized transform of $C^*(A)$, and hence exhibit an isomorphic copy of $C^*(A)$.

Theorem 3. *Let $A = N \oplus \hat{P}$ be a quasinormal operator. Let $\mathfrak{D}$ be the set of functions f on Q such that*

(1) f restricted to Q_1 is complex valued and f restricted to Q_2 takes values in $C^*(U_+)$;

(2) the functions $f: Q_1 \to \mathbf{C}$ and $f: Q_2 \to C^*(U_+)$ are continuous with the norm topology on the image space,

(3) if $0 \neq z \in a(\hat{P})$ then $f(z, 0) = (z|z|^{-1})\hat{}\,(f(0, |z|))$; and

(4) if (x_n) is a sequence in $\mathrm{sp}(P)$ which converges to zero, then $f(0, x_n)$ converges to $f(0, 0)I$ in norm.

Then $\mathfrak{D}$ is the regularized transform of $C^*(A)$ and is a C^*-algebra which is isomorphic to $C^*(A)$.

Proof. Let $\mathcal{C}$ be the set of functions on Q which satisfy properties (1), (2), and (4). Then $\mathcal{C}$ is clearly a maximal full algebra of operator fields on Q, and $\mathfrak{X}$ is clearly in $\mathcal{C}$, hence $\{\widetilde{B}: B \in C^*(A)\}$ is a full algebra of operator fields contained in $\mathcal{C}$. For $0 \neq z \in a(\hat{P})$ let R_z be the $(\mathcal{C}(z, 0), \mathcal{C}(0, |z|))$ correlation determined by the identity map of the scalars onto the scalars and the map $(z|z|^{-1})\hat{}$ of $C^*(U_+)$ onto the scalars. Then $\mathcal{C}((z, 0), (0, |z|); R_z)$ is the set $\{f \in \mathcal{C};\ f(z, 0) = (z|z|^{-1})\hat{}\,(f(0, |z|))\}$. It is easy to see that $\mathfrak{X} \in \mathcal{C}((z, 0), (0, |z|); R_z)$; hence by Fell's Stone-Weierstrass theorem, we will have $\mathfrak{D} = \{\widetilde{B}: B \in C^*(A)\}$ if we show that the R_z, $0 \neq z \in a(\hat{P})$ are the only correlations containing $\{\widetilde{B}: B \in C^*(A)\}$. First, suppose that R is a $(\mathcal{C}(0, x_1), \mathcal{C}(0, x_2))$ correlation with $0 \neq x_1 \in \mathrm{sp}(P)$, $0 \neq x_2 \in \mathrm{sp}(P)$, and $\mathcal{C}((0, x_1), (0, x_2); R)$ containing $\{\widetilde{B}: B \in C^*(A)\}$. Then there is a C^*-algebra $\mathfrak{R}$ and *-homomorphisms F and G of $C^*(U_+)$ onto $\mathfrak{R}$ which determine R. But then $F(\widetilde{A^*A}(0, x_1)) = G(\widetilde{A^*A}(0, x_2))$ and $\widetilde{A^*A}(0, x_1) = x_1^2 U_+^* U_+ = x_1^2 I$, so $x_1 = x_2$. Likewise, there are no $(\mathcal{C}(z, 0), \mathcal{C}(0, x))$ correlations with $z \in a(\hat{P}) \cup \mathrm{sp}(N)$, $0 \neq x \in \mathrm{sp}(P)$ and $|z| \neq x$. If R is a $(\mathcal{C}(x, 0), \mathcal{C}(y, 0))$ correlation with $x, y \in a(\hat{P}) \cup \mathrm{sp}(N)$ and if R is determined by *-homomorphism F and G onto some C^*-algebra $\mathfrak{R}$, then we have $F(\mathfrak{X}(x, 0)) = G(\mathfrak{X}(y, 0))$, so $x = y$. Finally, if R is a $(\mathcal{C}(z, 0), \mathcal{C}(0, |z|))$ correlation with $0 \neq z \in a(\hat{P})$ and R is determined by *-homomorphisms F and G of $\mathbf{C}$ and $C^*(U_+)$, respectively, onto some C^*-algebra $\mathfrak{R}$ then $F(\mathfrak{X}(z, 0)) = G(\mathfrak{X}(0, |z|))$, so $zI = |z|G(U_+)$ and $G(U_+) = (z|z|^{-1})\hat{}\,(U_+)$, so $G = (z|z|^{-1})\hat{}$. Thus $R = R_z$. Hence the correlations R_z, $0 \neq z \in a(\hat{P})$ are the only correlations which contain $\{\tilde{B}: B \in C^*(A)\}$, and it follows that $\mathfrak{D} = \{\tilde{B}: B \in C^*(A)\}$.

We remark that Theorem 3 can be used to show that the C^*-algebras generated by any two nonunitary isometries are isomorphic, a result that Coburn proved in [7]. For if V is a nonunitary isometry then $V = \hat{I}_\alpha \oplus W$, where I_α is the identity on some α-dimensional Hilbert space, and W is unitary. Then the set Q in Theorem 3 is $\{(z, 0): |z| = 1\} \cup \{(0, 1)\}$, and it is easily seen that $C^*(V)$ is *-isomorphic to $C^*(U_+)$.

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